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NUMERICAL RADII OF OPERATOR MATRICES AND ITS APPLICATION

(作用素行列の数域半径とその応用)

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ABSTRACT. Garcia gave an upper bound of $\|A+iB\|$ for self-adjoint operators A and B. We give a matrix representation of $\|A+iB\|$ and a generalization of Garcia's result. For it we give a numerical range of $\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix}$ for an operator T and real numbers a and b. On the other hand, Furuta gave a numerical range of $S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$ for an operator T and nonnegative real numbers a, b, c and d. We pointed out $w(S) = w(\operatorname{Re} S)$ under the condition that a, b, c and d are real numbers with $cd \geq 0$.

1. INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H. The numerical range $W(T)$ of an operator T is defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

Toeplitz-Hausdorff's theorem implies that the numerical range $W(T)$ is a convex set on the complex plane (cf. [3]). Moreover the numerical radius $w(T)$ of an operator T is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}.$$

It is known that $w(T) \leq \|T\|$, and $w(T) = \|T\|$ for normal operators T.

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In [4], Garcia showed the following theorem:

Theorem A. *If A and B are self-adjoint operators with $m \leq A \leq M$, then*

$$\|A+iB\| \leq \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|B\|^2}. \tag{1.1}$$

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The norm $\|A+iB\|$ is represented by an operator matrix as follows: $\|A+iB\| = \left\|\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\right\|$ (see Lemma 3.1). So the inequality (1.1) is rewritten as follows:

$$(1.2)\quad w\left(\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\right) = \left\|\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\right\| \le \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|B\|^2}.$$

In this note, we shall calculate $w\left(\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix}\right)$ for an operator T and real numbers a and b (see Theorem 2.2). As a result, we give an upper bound of $w\left(\begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix}\right)$ for an operator T and self adjoint operators A_1, A_2 as a generalization of (1.2) (i.e., (1.1)) (see Theorem 3.2). For it, the following equation plays an elementary and essential role

$$w\left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right) = \left\|\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right\| = \sqrt{a^2 + \|T\|^2}$$

for an operator T and $a \in \mathbb{R}$ (see Lemma 2.1(ii)).

On the other hand, Furuta [2] gave the numerical radius $w(S)$ of S which is defined by (1.3). We show that it is sufficiently to obtain the value of $w(\operatorname{Re} S)$ $(= w(\frac{S+S^*}{2}))$ for the calculation of $w(S)$ by using Theorem 2.2.

Theorem B. *Let*

$$(1.3)\qquad S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$$

be an operator on a Hilbert space $H = H_1 \oplus H_2$ where T is an operator from H_2 to H_1 and a, b, c and d are nonnegative real numbers. Then

$$(1.4)\qquad w(S) = \frac{1}{2}(a+b) + \frac{1}{2}\sqrt{(a-b)^2 + (c+d)^2\|T\|^2}.$$

We calculate $w(S)$ under the condition $a, b, c, d \in \mathbb{R}$ with $cd \ge 0$ (see Theorem 3.3).

2. Numerical radius for self-adjoint operators

For an operator T and a real number a, we give some properties of $\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}$ to generalize (1.2) (i.e., (1.1)) in the following lemma:

Lemma 2.1. *Let T be an operator and a be a real number. Then the following holds:*
(i) $W\left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right)$ *is symmetric, i.e.,*

$$\alpha \in W\left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right) \iff -\alpha \in W\left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right),$$

(ii) $w\left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right) = \left\|\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}\right\| = \sqrt{a^2 + \|T\|^2},$

(iii) $\overline{W}\left(\left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\right) = \left[-\sqrt{a^2+\|T\|^2}, \sqrt{a^2+\|T\|^2}\right]$

where $\overline{W}(T)$ is a closure of $W(T)$.

Proof. (i) For $\alpha \in W\left(\left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\right)$, we have

$$\alpha = \left\langle \left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right), \left(\begin{array}{c} x \\ y \end{array}\right)\right\rangle = a\,\|x\|^2 + 2\,\mathrm{Re}\,\langle Ty, x\rangle - a\,\|y\|^2$$

for some unit vector $\left(\begin{array}{c} x \\ y \end{array}\right)$.

On the other hand, since $\left(\begin{array}{c} \frac{\|y\|}{\|x\|}x \\ -\frac{\|x\|}{\|y\|}y \end{array}\right)$ is a unit vector, we have

$$\left\langle \left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\left(\begin{array}{c} \frac{\|y\|}{\|x\|}x \\ -\frac{\|x\|}{\|y\|}y \end{array}\right), \left(\begin{array}{c} \frac{\|y\|}{\|x\|}x \\ -\frac{\|x\|}{\|y\|}y \end{array}\right)\right\rangle = a\|y\|^2 - 2\,\mathrm{Re}\langle Ty, x\rangle - a\|x\|^2 = -\alpha$$

and $\left\langle \left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\left(\begin{array}{c} \frac{\|y\|}{\|x\|}x \\ -\frac{\|x\|}{\|y\|}y \end{array}\right), \left(\begin{array}{c} \frac{\|y\|}{\|x\|}x \\ -\frac{\|x\|}{\|y\|}y \end{array}\right)\right\rangle \in W\left(\left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\right)$. Hence we have $-\alpha \in W\left(\left(\begin{array}{cc} aI & T \\ T^* & -aI \end{array}\right)\right)$.

(ii) Since $\left(\begin{array}{cc} aI & T^* \\ T & -aI \end{array}\right)$ is self-adjoint, we have

$$\left\|\left(\begin{array}{cc} a & T \\ T^* & -a \end{array}\right)\right\|^2 = \left\|\left(\begin{array}{cc} a & T \\ T^* & -a \end{array}\right)^2\right\| = \left\|\left(\begin{array}{cc} a^2+TT^* & 0 \\ 0 & a^2+T^*T \end{array}\right)\right\| = a^2 + \|T\|^2.$$

(iii) is obvious by (i) and (ii). □

In the following theorem, we give the numerical radius of the self-adjoint operator matrix $\left(\begin{array}{cc} aI & T \\ T^* & bI \end{array}\right)$:

Theorem 2.2. *Let $H = H_1 \oplus H_2$ be a Hilbert space and let T be an operator from H_2 to H_1. Let a and b be real numbers. Then*

$$w\left(\left(\begin{array}{cc} aI & T \\ T^* & bI \end{array}\right)\right) = \left\|\left(\begin{array}{cc} aI & T \\ T^* & bI \end{array}\right)\right\| = \frac{1}{2}|a+b| + \frac{1}{2}\sqrt{(a-b)^2 + 4\,\|T\|^2}. \tag{2.1}$$

Proof. The first equality is obviously. Next we have

$$\begin{aligned} w\left(\left(\begin{array}{cc} aI & T \\ T^* & bI \end{array}\right)\right) &= w\left(\frac{1}{2}(a+b) + \left(\begin{array}{cc} \frac{a-b}{2}I & T \\ T^* & -\frac{a-b}{2}I \end{array}\right)\right) \\ &= \frac{1}{2}|a+b| + w\left(\left(\begin{array}{cc} \frac{a-b}{2}I & T \\ T^* & -\frac{a-b}{2}I \end{array}\right)\right) \quad \text{by Lemma 2.1 (iii)} \\ &= \frac{1}{2}|a+b| + \sqrt{\left(\frac{a-b}{2}\right)^2 + \|T\|^2} \quad \text{by Lemma 2.1 (ii)} \end{aligned}$$

$$= \frac{1}{2}|a+b| + \frac{1}{2}\sqrt{(a-b)^2 + 4\|T\|^2}$$

and hence the second equality holds. □

3. Main results

The following lemma gives matrix representation of $\|A + iB\|$ for self-adjoint operators A and B.

Lemma 3.1. *Let X and Y be operators on H. Then*

$$\left\| \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\| = \left\| \begin{pmatrix} X & iY \\ iY & X \end{pmatrix} \right\| = \max\{\|X+iY\|, \|X-iY\|\}. \tag{3.1}$$

In particular, if A and B are self-adjoint operators, then

$$\left\| \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right\| = \left\| \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right\| = \left\| \begin{pmatrix} A & B \\ iB & A \end{pmatrix} \right\| = \|A+iB\| = \|A-iB\|. \tag{3.2}$$

Proof. We only prove the equality (3.1). Let I be the identity operator of H. Then we have

$$\begin{aligned} \begin{pmatrix} X+iY & 0 \\ 0 & X-iY \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} X & iY \\ iY & X \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}. \end{aligned}$$

Since matrix operators $\frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ are unitary, we have

$$\begin{aligned} \left\| \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\| = \left\| \begin{pmatrix} X & iY \\ iY & X \end{pmatrix} \right\| &= \left\| \begin{pmatrix} X+iY & 0 \\ 0 & X-iY \end{pmatrix} \right\| \\ &= \max\{\|X+iY\|, \|X-iY\|\}. \end{aligned}$$

So the desired equalities hold. □

From (3.2), (1.1) in Theorem A can be interpreted as

$$w\left(\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\right) \le \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|B\|^2}. \tag{3.3}$$

Thus the following theorem is a generalization of Theorem A:

Theorem 3.2. *Let T be an operator from H_2 to H_1, and let A_i be self-adjoint operators on H_i with $\sigma(A_i) \subset [m, M]$ where $\sigma(A_i)$ is the spectrum of A_i $(i = 1, 2)$. Then*

$$w\left(\begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix}\right) \le \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|T\|^2}. \tag{3.4}$$

Proof. Since $m \le A_i \le M$ $(i = 1, 2)$, we have

$$\begin{pmatrix} m & T \\ T^* & -M \end{pmatrix} \le \begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix} \le \begin{pmatrix} M & T \\ T^* & -m \end{pmatrix}.$$

Here we have by Theorem 2.2

$$\left\| \begin{pmatrix} M & T \\ T^* & -m \end{pmatrix} \right\| = \left\| \begin{pmatrix} m & T \\ T^* & -M \end{pmatrix} \right\| = \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\,\|T\|^2}.$$

Hence the desired inequality (3.4) holds. □

Next, we pay attention to the fact that $w(S) = w(\mathrm{Re}\, S)$ for S in (1.3). As a consequence, Theorem 2.2 is applicable to the following theorem. It is a generalization of Theorem B.

Theorem 3.3. *Let $H = H_1 \oplus H_2$ be a Hilbert space and let T be an operator from H_2 to H_1. Let a, b, c and d be real numbers with $cd \ge 0$. Suppose that $S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$ be an operator on H. Then*

$$w(S) = w(\mathrm{Re}\, S) = \frac{1}{2}|a + b| + \frac{1}{2}\sqrt{(a - b)^2 + (c + d)^2\,\|T\|^2}. \tag{3.5}$$

Proof. We have

$$\begin{aligned} w(S) &= \sup\left\{ \left| \left\langle \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \right| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup\left\{ \left| a\|x_1\|^2 + b\|x_2\|^2 + c\langle Tx_2, x_1\rangle + d\overline{\langle Tx_2, x_1\rangle} \right| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup\left\{ \left| a\|x_1\|^2 + b\|x_2\|^2 \right| + |(c + d)\langle Tx_2, x_1\rangle| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\}. \end{aligned} \tag{3.6}$$

The last equality of (3.6) is ensured by $cd \ge 0$. Moreover we have

$$\begin{aligned} w(\mathrm{Re}\, S) &= \sup\left\{ \left| \left\langle \begin{pmatrix} aI & \frac{c+d}{2}T \\ \frac{c+d}{2}T^* & bI \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \right| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup\left\{ \left| a\|x_1\|^2 + b\|x_2\|^2 + (c + d)\,\mathrm{Re}\langle Tx_2, x_1\rangle \right| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup\left\{ \left| a\|x_1\|^2 + b\|x_2\|^2 \right| + |(c + d)\langle Tx_2, x_1\rangle| \; ; \; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\}. \end{aligned}$$

So the first equality of (3.5) holds.

Next, replacing $\frac{c+d}{2}T$ to T in Theorem 2.2, we have

$$w(\mathrm{Re}\, S) = w\left(\begin{pmatrix} aI & \frac{c+d}{2}T \\ \frac{c+d}{2}T^* & bI \end{pmatrix} \right) = \frac{1}{2}|a + b| + \frac{1}{2}\sqrt{(a - b)^2 + (c + d)^2\,\|T\|^2},$$

and hence the second equality holds. □

If $cd < 0$ in Theorem 3.3, then the last equation of (3.6) and so the first equality of (3.5) be not ensured. We confirm this result by using 2×2 real matrix $S = \begin{pmatrix} v & -w \\ w & v \end{pmatrix}$

where $v, w \neq 0$ and $v^2 + w^2 = 1$. Since S is unitary, we have $w(S) = 1$. On the other hand, it follows from $\operatorname{Re} S = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ that $w(\operatorname{Re} S) = |v| (< 1)$.

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